# Topological Gravity Localization on a $\delta$-function like Brane

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Besides the String Theory context, the quantum General Relativity can be studied by the use of constrained topological field theories. In the celebrated Plebanski formalism, the constraints connecting topological field theories and gravity are imposed in space-times with trivial topology. In the braneworld context there are two distinct regions of the space-time, namely, the bulk and the braneworld volume. In this work we show how to construct topological gravity in a scenario containing one extra dimension and a $\delta$-function like 3-brane which naturally emerges from a spontaneously broken discrete symmetry. Starting from a $D = 5$ theory we obtain the action for General Relativity in the Palatini form in the bulk as well as in the braneworld volume. This result is important for future insights about quantum gravity in brane scenarios.

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Physics of extra dimensions has been successful. Indeed, some of the crucial drawbacks of the Standard Model are better understood when the theories are embedded in $D > 4$ space-times. In the context of String Theory and Branes, we have at hand a rather general and simple view of how a theory of grand unification should be constructed [1]. In Braneworld models, the Standard Model is completely embedded in scenarios containing extra dimensions and intersecting branes [2]. In particular, the Randall-Sundrum scenario [3] gives an elegant solution to an old problem of the Standard Model: the gauge hierarchy problem. Besides, we can understand how the gravitational field can be trapped to the 4-dimensional world. Extensions of the Randall-Sundrum model in order to study supersymmetry [4] have been made, including in the context of String Theory [5].

The results of the Randall-Sundrum model are based in the classical theory of General Relativity, a theory that possesses as a fundamental quantity the space-time metric $g_{\mu\nu}$. However, as is well known, the Einstein-Hilbert formulation of gravitation does not support the common procedures of quantization. In other words, the quantum theory of gravity suffer problems of non-renormalizability. In order to avoid this bad characteristic new methods were developed [6]. Such approaches are based on topological gravity theories, i. e., theories where the metric $g_{\mu\nu}$ is derived from more fundamental quantities. In the context of stringy models, the idea of background independence is very important in order to fully understand the dynamics of space-time [7].

In view of these developments, it is interesting to build and study topological gravity models in the context of extra dimensions and branes. In this direction, some discussions have already been made. Boyarsky and Kulik [8] have studied topological gravity on a singular boundary brane from a higher derivative topological theory in order to link their model with solitonic brane backgrounds. On the other hand a BF topological theory which induces gravity on a boundary through specific conditions was discussed by Henty [9]. In particular recently we have proposed an approach to the gauge hierarchy problem from the topological gravity viewpoint [10]. In that case, no mention is made about gravity in the bulk $D = 5$ space-time.

In this letter, we show how to construct topological gravity in a scenario containing one infinity extra dimension and a $\delta$-function like brane. In the Plebanski formalism, the link between a Topological Field Theory and General Relativity is made via constraints expressed by Lagrange multiplier fields $\underset{\sim}{\Phi}$. The important detail is that the constraints are valid for a space-time without any mention to a boundary. In our case, we have two distinct regions of the space-time: the $D = 5$ bulk and the $D = 4$ domain hyperplane (3-brane). Where should we implement the constraints? The answer is that if we impose the constraints in $D = 5$ bulk, then we can build up the Palatini action in the bulk space-time and also in the 3-brane after a field identification.

Topological gravity can be studied by means of constrained topological field theories. Here we follow the conventions introduced by Freidel *et al.* [11]. Thus, we regard Greek characters as space-time indices, Latin letters are internal indices, and a tilde over the fields represent the fact that they are tensorial densities. Our model is based in the following action in five dimensions:

$$S = \int dx^5 \left[ \widetilde{B}^{\mu\nu}_{ij} F^{ij}_{\mu\nu} + \frac{1}{2} \underset{\sim}{\Phi}_{m\mu\nu\rho\sigma} \epsilon^{mijkl} \widetilde{B}^{\mu\nu}_{ij} \widetilde{B}^{\rho\sigma}_{kl} + L_{brane} + k \partial_\alpha \varphi \widetilde{C}^{\mu\nu\alpha}_{ij} F^{ij}_{\mu\nu} \right], \tag{1}$$

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where $L_{brane} = \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - V(\varphi)$. This action is a functional of SO$(4,1)$ gauge fields $A_\mu^{ij}$, bivector fields $\widetilde{B}_{ij}^{\mu\nu}$, the fields $\widetilde{C}_{ij}^{\mu\nu\alpha}$, Lagrange multiplier fields $\underset{\sim}{\Phi}_{m\mu\nu\rho\sigma}$ and a real scalar field $\varphi$. The two first terms define the Plebanski action in $D=5$. The second and third terms represent the part of the action that generates the 3-brane of this model. Indeed, this 3-brane is a domain wall embedded in a $D=5$ space-time. For this, we suppose $\varphi=\varphi(x_4)$ and use $V(\varphi)=\lambda(1-\cos\varphi)$. The last term in Eq.(1) is a topological term that will give an effective BF type action over the 3-brane in $D=4$. It is just this term the responsible for topological gravity on the brane.

Having made the first presentations, it is now important to discuss the symmetries of this model. The first one is a discrete symmetry: the action is invariant under the change $\varphi\rightarrow\varphi+2\pi$ (it is like a Peccei-Quinn symmetry). This symmetry is important because, if spontaneously broken, it will give us kink like defects. The defect related to the topological sector containing just one soliton will be the 3-brane of the model (the scalar field model treated here is the sine-Gordon one that, as it is well known, has several solitonic solutions). The second type of symmetry is the general coordinate transformation. In this case, this symmetry plays the role of a SO$(4,1)$ gauge symmetry. Despite the terms responsible for the brane, the action is generally covariant: the fields $\widetilde{B}_{ij}^{\mu\nu}$ and $\widetilde{C}_{ij}^{\mu\nu\alpha}$ scales as tensor densities of weight one, while the multipliers scale as tensorial densities of weight minus one (represented as a single tilde below the symbol '$\Phi$'). The role played by the 3-brane it is to break this SO$(4,1)$ symmetry in order to induce an SO$(3,1)$ symmetry over itself. This fact is not a surprise. In string theories, for instance, this is a basic characteristic of models containing D-branes.

Before study the main idea of this work, we review quickly some aspects of the Plebanski formulation of topological gravity. More rigorous details can be found in Ref. [11] and references therein. The fields $\widetilde{B}_{ij}^{\mu\nu}$ and $\widetilde{C}_{ij}^{\mu\nu\alpha}$ are defined in the following manner:

$$\widetilde{B}_{ij}^{\mu\nu} = \frac{1}{4}\tilde{\epsilon}^{\mu\nu\alpha\beta\lambda}B_{\alpha\beta\lambda ij} \tag{2}$$

and

$$\widetilde{C}_{ij}^{\mu\nu\alpha} = \frac{1}{12}\tilde{\epsilon}^{\mu\nu\alpha\beta\lambda}C_{\beta\lambda ij}. \tag{3}$$

In order to write an action for gravity we must compute the variation of the action (1) with the Lagrange multiplier field $\underset{\sim}{\Phi}$. For such, we have to postulate the following property obeyed by the Lagrange multiplier:

$$\epsilon^{m\mu\nu\rho\sigma}\underset{\sim}{\Phi}_{m\mu\nu\rho\sigma} = 0. \tag{4}$$

Therefore, the variation of the action (1) give us

$$\frac{\delta\underset{\sim}{\Phi}_{m\mu\nu\rho\sigma}}{\delta\underset{\sim}{\Phi}_{n\alpha\beta\gamma\lambda}}\epsilon^{mijkl}\widetilde{B}_{ij}^{\mu\nu}\widetilde{B}_{kl}^{\rho\sigma} = 0. \tag{5}$$

Comparing the variation of Eq. (4) with the Eq. (5) we obtain, for some coefficients $c_\alpha^m$, that

$$\epsilon^{mijkl}\widetilde{B}_{ij}^{\mu\nu}\widetilde{B}_{kl}^{\rho\sigma} = c_\alpha^m\epsilon^{\alpha\mu\nu\rho\sigma}. \tag{6}$$

The coefficients $c_\alpha^m$ then satisfy:

$$c_\alpha^m = \frac{1}{5!}\epsilon_{\alpha\mu\nu\rho\sigma}\epsilon^{mijkl}\widetilde{B}_{ij}^{\mu\nu}\widetilde{B}_{kl}^{\rho\sigma}. \tag{7}$$

The major importance of Eq. (7) is viewed in the following theorem introduced by Feidel *et. al.* [11]:

*Theorem: In $D>4$, a generic B field satisfies the constraints of Eq. (7) if and only if it comes from a frame field. In another words, if B is non-degenerate and satisfies Eq. (7), then there are frame fields $e_i^\mu$ such that

$$\widetilde{B}_{ij}^{\mu\nu} = \pm|e|e_i^{[\mu}e_j^{\nu]}. \tag{8}$$

In the last equation, $|e|$ is the determinant of the frame field $e_i^\mu$. Now, is important to note that the constraints of Eq. (7) give 5-dimensional $e_i^\mu$ fields, i.e., we are imposing constraints in the bulk space-time. Let us see what are the consequences for the physics on the brane. We must consider that in the vicinity of the brane worldsheet the metric may be written in terms of coordinates $\xi^a$ and $w$ ($w$ is the normal distance from the hyperplane) in the following way:

$$ds^2 = \gamma_{ab}d\xi^a d\xi^b - dw^2. \tag{9}$$

In the last equation $\gamma_{ab} = g_{\mu\nu} x^{\mu}_{,a} x^{\nu}_{,b}$ is the worldsheet induced metric (this metric is constructed with the frame fields encountered in the theorem above). The equation of motion for the $\varphi$ field, regarded as static and having only dependence in the $x^5 \equiv w$ coordinate ($\varphi = \varphi(w)$), is given by:

$$-\frac{d^2\varphi}{dw^2} + \frac{dV(\varphi)}{d\varphi} - \frac{d}{d\varphi}(k\widetilde{C}^{\mu\nu 4}_{ij} F^{ij}_{\mu\nu}) = 0. \tag{10}$$

Putting $V(\varphi) = \lambda(1 - \cos\varphi)$ and discarding fluctuations due to the fields $\widetilde{C}^{\mu\nu\alpha}_{ij}$ and $F^{ij}_{\mu\nu}$ we conclude that

$$-\frac{d^2\varphi}{dw^2} + \lambda \sin\varphi = 0, \tag{11}$$

which solution is the solitonic 3-brane of this model (in this step, the symmetry $\varphi \rightarrow \varphi + 2\pi$ is spontaneously broken, favoring the appearance of domain hyperplanes), namely:

$$\varphi(w) = 4tg^{-1} exp\left(\sqrt{\lambda} w\right) \tag{12}$$

In contrast with present works in topological gravity where the brane is introduced *ad hoc*, it is worth mentioning that here the 3-brane appears from a very natural physical way.

Let us return to the action (1) and see its last term, i.e.,

$$S_{brane} \sim k \int d^5x \left[\frac{d\varphi(w)}{dw} \widetilde{C}^{\mu\nu 4}_{ij} F^{ij}_{\mu\nu}\right]. \tag{13}$$

In the thin hyperplane limit we have $\frac{d\varphi(w)}{dw} \sim \delta(w)$. Then, from Eq. (13) we obtain an effectively 4-dimensional BF like action:

$$S_{4d} \mapsto k \int d^4x \left[\widetilde{C}^{\mu\nu 4}_{ij} F^{ij}_{\mu\nu}\right]. \tag{14}$$

The interpretation is that, regarding the conditions discussed, some sort of BF theory should describe the physics in the 4-dimensional world. It is convenient to note that the breaking of the discrete symmetry of this model induces the explicit break of the SO$(4,1)$ gauge symmetry down to a SO$(3,1)$ symmetry in the brane. The Greek and Latin indices are enumerated from 0 till 4 in $D = 5$. By construction (see Eq. (3)), if we are studying a 4-dimensional world, the indices match. Now, the most important thing. If we identify the degrees of freedom of $\widetilde{C}^{\mu\nu 4}_{ij}$ with that coming from Eq. (8), i.e., if

$$\widetilde{C}^{\mu\nu 4}_{ij}|_{D=4} \equiv \widetilde{B}^{\mu\nu}_{ij} = \pm|e^{(5)}|e^{[\mu}_{i} e^{\nu]}_{j}|_{D=5}, \tag{15}$$

then we can substitute this result back in the original action (Eq. (1)) and rewrite it effectively as

$$S_{eff.} = \pm \int d^5x \left[|e^{(5)}|e^{[M}_{I} e^{N]}_{J} F^{IJ}_{MN} + k'\delta(w)|e^{(4)}|e^{[\mu}_{i} e^{\nu]}_{j} F^{ij}_{\mu\nu}\right]. \tag{16}$$

Here we have written capital Latin indices in order to make distinction between $D = 5$ and $D = 4$ internal and space-time indices. The result is just the Palatini action for $D = 5$ gravity together a new term. This new term is the Palatini action for gravity in a 3-brane of the $\delta$-function type. The resemblance between this result and that obtained by Dvali *et. al.* [12] should be noted. In that work the action (16) can be obtained by loop corrections due to localized matter on the brane or due to specific couplings between gravity and scalar fields. Finally, a simple detail should be considered. The identification procedure established by Eq. (15) depends specifically on $|e^{(5)}|$. Fortunately, a 4-dimensional determinant may be constructed from it. Here, we must consider, for instance, that the new constant $k'$ in Eq. (16) depends on the extra dimension. This indeed may be of great importance in order to understand the physical effects of the extra dimension in $D = 4$ in the context of topological gravity.

The main conclusion of this work is that we can construct a $\delta$-like brane scenario in the context of topological gravity by the use of a topological constrained field theory. On the other hand, we use a spontaneously breaking symmetry mechanism in order to generate the 3-brane, which it is a new approach in this context. The procedure introduced here depends exclusively on constraints imposed on the bulk $D = 5$ space-time. Then, we obtain gravity theory in $D = 4$ starting from a very specific interaction in the action (1). In discussions about the *physical reality* of the formalism of topological gravity, the non-dynamical nature of the degrees of freedom in this kind of theory is often cited. In this

regards, the model described here looks like some others gravity localization approaches. In these approaches, some *physical* gravitational degrees of freedom are effectively trapped to a 3-brane. However, in the context of topological gravity, it is hard to affirm if such a mechanism could happen or not. The unique characteristics that guarantee the existence of a 4-dimensional theory in the brane are mathematical: $\mathrm{SO}(3,1)$ gauge symmetry $\Rightarrow D = 4$ theory.

An interesting question is the cosmological constant. A generalization of the Plebanski action results in the Palatini action for general relativity plus a cosmological constant term [13]. In the lines discussed here, the generation, if possible, of cosmological constant in the brane should be investigated. In the context of topological gravity, another interesting subject is the Immirzi parameter [14] and the related consequences within a brane formalism.

A natural further step would be an attempt to build up a topological version of the Randall-Sundrum scenario [3]. The motivations behind this step are several: the possibility of quantization of the scenario; alternatives to solve the standard model problems such as the hierarchy problem, etc.

The importance of our result lies in the fact that it is possible to fully quantize topological gravity models. To the best of our knowledge, physical consequences related to the existence of extra dimensions have been well formulated in the context of classical theories, not in the context of quantum models. The quantum consequences are welcome and interesting to be studied because of at least two good reasons: i) important and modern results (like that of the Randall-Sundrum scenario) could be reinforced and ii) we may or may not decide in favor of the formalisms of topological gravity. We believe that our result should help in future attempts to quantize gravity in brane backgrounds.

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